**Sequences & Series – Notes**

**Recursive rule for arithmetic progressions**: Tn+1 = Tn + d

**General rule for arithmetic progressions**: Tn = a + (n–1)d

Where: d = Common difference.

n = Term number.

**Recursive rule for geometric progressions**: Tn+1 = r x Tn

**General rule for geometric progressions**: Tn = T1 x rn-1

 Where: r = Common ratio.

The difference between any 2 consecutive terms is constant → arithmetic sequence.

The ratio between any 2 consecutive terms is constant → geometric sequence.

**Arithmetic sequences**: Sn = $\frac{n}{2}$(2a + (n–1)d)

 Sn = $\frac{n}{2}$(a + l) Where: l = Last term in sequence.

**Geometric sequences**: Sn = $\frac{a(1-r^{n})}{1-r}$ –1 ≤ r < 1

 Sn = $\frac{a(r^{n}-1)}{r-1}$ r ≠ 1

**Sum to infinity**: S∞ = $\frac{a}{1-r}$ –1 < r < 1

**Tn = Sn – Sn-1**

**Sn = rn+1 – r**

**Arithmetic Progressions**

**An arithmetic sequence is described by the rule Tn+1 = Tn + 6 with T1 = –96.**

**[a] Find the general rule of this sequence in the form Tn = a + bn.**

Tn = –96 + (n–1)6 = 6n – 102

**The sum of the first n terms of an arithmetic progression is given by Sn = 3n2 – 21n. Find:**

**[a] The first 3 terms of the sequence.**

S1 = T1 = 3 – 21 = –18

S2 = 3(2)2 – 21(2) = –30 → S2 = S1 + T2 → T2 = S2 – S1 = –30 – (–18) = –12

S3 = 3(3)2 –21(3) = –36 → S3 = S2 + T3 → T3 = S3 – S2 = –36 – (–30) = –6

**[b] The recursive rule of the sequence.**

Tn+1 = Tn + 6 with T1 = –18

**[c] The sum of all terms between the 11th term and the 20th term inclusive.**

Sum = S20 – S10 = 3(20)2 – 21(20) – (3(10)2 – 21(10)) = 780 – 90 = 690

**The 8th term and 12th term of an arithmetic sequence are 24 and 40 respectively. Find the recursive rule for the sequence.**

Common difference = $\frac{40-24}{4}$ = 4

T1 = 24 – 4 x 7 = –4 → Tn+1 = Tn + 4 with T1 = –4

**The 6th term of an arithmetic sequence is double its 4th term. The first term of the sequence is 20 and the common difference is d.**

**[a] Show that T4 = 2 x d.**

T6 = 2 x T4 = T4 + 2 x d → T4 = 2d

**[b] Hence, find the general rule for the sequence.**

T4 = 2d = 20 + 3d → d = –20

Tn = 20 + (n–1)(–20) = 20 – 20n + 20 = 40 – 20n

**A grandfather clock makes as many long chimes as the hour of the day, on the hour, every hour. Assume 12-hour time.**

**[a] How many long chimes would this clock make in a 24 hour day?**

S12 = $\frac{12}{2}$(2(1) + (12 – 1)1) = 78 → 2S12 = 2 x 78 = 156

**In addition, this clock makes 1 short chime to mark the passage of the first quarter of any hour, 2 short chimes to mark the passage of the first half-hour of any hour and 3 short chimes to mark the passage of the third quarter of any hour.**

**[b] How long after 12 noon would it take for the clock to have made a total of 105 chimes (long and short)? Justify your answer.**

Sn = $\frac{n}{2}$(2(1) + (n – 1)1) + (1+2+3)n= 105 → CAS solve → n = 9.38 → ~9 hours and 45 minutes

**Geometric Progressions**

**$1 000 000 is invested in an account that pays interest at a rate of 5% per annum compounded annually. Let B(n) be the account balance at the end of n years.**

**[a] Find the general rule for the account balance at the end of n years.**

Tn = 1 000 000 x 1.05n

**[b] Find the average percentage growth rate in the first 10 years.**

T10 = 1 000 000 x 1.0510 = 1628894.627

Growth = 1628894.627 – 1 000 000 = 628894.627

Percentage growth rate = $\frac{628894.627}{1 000 000}$ x 100 = 62.89%

Average percentage growth rate = $\frac{62.89}{10}$ = 6.29%

**[c] Calculate the average percentage growth rate in the first 20 years.**

T20 = 1 000 000 x 1.0510 = 2653297.705

Growth = 2653297.705 – 1 000 000 = 1653297.705

Percentage growth rate = $\frac{1653297.705}{1 000 000}$ x 100 = 165.33%

Average percentage growth rate = $\frac{165.33}{20}$ = 8.27%

**[d] Give an explanation for the different answers in parts [b] and [c].**

Due to compounding effects, the annual growth increases from year to year. Hence, there’s proportionally a large increase over 20 years than 10 years and therefore a higher growth rate.

**To fight an infection, Steele has to take a course of medication which consists of 10 tablets to be taken over 10 days. One tablet is to be taken at the same time each day. Each tablet contains 50mg of a particular drug. At the end of each 24 hour period, only 20% of the drug remains in the body. The table below models the amount of drug in the body for a period of 5 days.**

|  |  |
| --- | --- |
| **Day**: | **Amount of drug in the body (mg):** |
| **Just before tablet is taken**: | **Just after tablet is taken**: |
| **1** | 0 | 50 |
| **2** | 50 x 0.2 | 50 + 50 x 0.2 |
| **3** | (50 x 0.2) + (50 x 0.22) | 50 +(50 x 0.2) + (50 x 0.22) |
| **4** | (50 x 0.2) + (50 x 0.22) + (50 x 0.23) | 50 + (50 x 0.2) + (50 x 0.22) + (50 x 0.23) |
| **5** | (50 x 0.2) + (50 x 0.22) + (50 x 0.23) + (50 x 0.24) | 50 + (50 x 0.2) + (50 x 0.22) + (50 x 0.23) + (50 x 0.24) |

**[a] Find the amount of drug left in the body just before the 6th tablet was taken.**

a = 10, r = 0.2 → S5 = $\frac{10(1-0.2^{5})}{1-0.2}$ = 12.496mg

**[b] Find the amount of drug in the body just after the 10th tablet is taken.**

a = 50, r = 0.2 → S9 = $\frac{50(1-0.2^{9})}{1-0.2}$ = 62.5mg

**[c] Find the amount of drug left in the body one week after the last tablet was taken.**

a = 62.5, r = 0.2 → T8 = 62.5 x 0.27 = 0.0008mg

**A group of engineers plan to bore a horizontal tunnel underneath a city to accommodate a railway line. The tunnel needs to be 3km long. Assume that the tunnel is a straight horizontal line. A tunnel boring machine is assembled and used to bore the tunnel. On the first 10 days, the machine bores a distance of 100m each day. Between the 11th and 20th day inclusive, because of the fragile nature of the environment, the distance bored each day is 90% of the distance bored in the previous day (the length bored is 90m on the 11th day). From and on the 21st day onwards, the machine bores an extra 10m more than what was bored on the previous day.**

**[a] Find the total distance bored in the first 10 days.**

Distance = 10 x 100 = 1000m = 1km

**[b] How much less distance is bored on the 20th day compared to the 19th day?**

a = 90, r = 0.9 → T20 = 90 x 0.99 = 34.86784401

 T19 = 90 x 0.98 = 38.7420489

Difference = 38.7420489 – 34.86784401 = 3.87m

**[c] Find the total distance bored in the first 20 days.**

a = 90, r = 0.9 → S10 = $\frac{90(1-0.9^{10})}{1-0.9}$ = $586.189403$9

Total distance = 1000 + $586.189403$9 = 1586.19m

**[d] How many days are required to complete boring the entire 3km tunnel?**

Distance on day 21 = 90(0.9)9 + 10 = 44.87m

Distance = 1000 + $\frac{90(1 – 0.9^{10})}{1 – 0.9}$ + $\frac{n}{2}$(2(44.87) + (n–1)10) = 3000

CAS solve → n = 13.29 → 14 days after day 20

Number of days = 20 + 14 = 34

**An investment account pays 15% interest compounded annually over a 15 year period. Brad invests $100 000 in this account for 15 years. No new money was added to and no withdrawals were made from the investment account.**

**[a] Calculate the value of the investment account after 10 years.**

a = 100 000, r = 1.15 → T10 = 100 000 x 1.1510 = $404555.77

**[b] Calculate the increase in value of the account during the 10th year.**

T10 – T9 = 100 000 x 1.1510 – 100 000 x 1.159 = $52768.14

**[c] Calculate the number of years required for the initial amount invested to double.**

100 000 x 1.15n = 200 000 → n = 4.96 = 5 years